On composition of symmetries

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1 Introduction

The symmetry group G of an equation on a vector space V is the group of transformations of V that are invariant to the solution set of the given equation. The aim of this paper is to investigate in general the properties of the composition $g_1, \ldots, g_n \in G$ of n elements in the symmetry group G acting on V.

Compositions of symmetry group elements can be described as induced by the symmetry algebra $\mathfrak{g}$ of G. This way the composition of symmetry action corresponds to the generalized commutator $[X_1, \ldots, X_n]$ of degree n of Lie algebra $\mathfrak{g}$ elements $X_1, \ldots, X_n$, which is an operation on the n-th tensor power of $\mathfrak{g}$. In order to characterize generalized commutators of symmetries of a given equation, we derive the Generalized infinitesimal symmetry criterion as a consequence of the Infinitesimal symmetry criterion for characterizing the symmetry algebra of an equation, giving necessary and sufficient condition for a generalized commutator acting as a symmetry of the equation. This way we derive the general system of partial differential equations that characterizes the composition of symmetry action.

Particular solutions to the problem are provided as examples using the Generalized Infinitesimal symmetry criterion in low dimensions for algebraic equations and ordinary differential equations.

2 Generalized symmetry criterion for algebraic equations

Here the simplified notation a_u means first order partial derivative of a with respect to u.

TODO one-parametric subgroup

TODO def invariant function

TODO def infinitesimal generator, symmetry algebra

Statement 2.1 *Let G be a connected group of transformations acting on the manifold M. A smooth real-valued function $f : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$v(f) = 0,$$

for all $x \in M$ and every infinitesimal generator v of G.

Statement 2.2 (Infinitesimal invariance criterion for algebraic equations) *Suppose*

$$F^k(x) = 0$$

for $k = 1, \dots, d$ is a system of algebraic equations of maximal rank defined over M. If G is a connected local Lie group of transformations acting on M then G is a symmetry group of the system if and only if

$$vF^k(x) = 0,$$

where $k = 1, \dots, d$ whenever

$$F(x) = 0,$$

for every infinitesimal generator v of G,

Theorem 2.1 (Generalized infinitesimal invariance criterion for algebraic equations) *Let G be a connected Lie group of transformations acting on the n-dimensional manifold M. Let $F : M \to \mathbb{R}^d$, $d \leq n$ define a system of algebraic equations*

$$F^k(x) = 0$$

for $k = 1, \dots, d$ and assume the system is of maximal rank, meaning the Jacobian matrix $(\frac{\partial F^k}{\partial x_i})$ is of rank d at every solution x of the system. Then G is a q-symmetry group of the system if and only if

$$vF^k(x) = 0,$$

where $k = 1, \dots, d$ and $v = \mathrm{Alt}(v_1, \dots, v_q)$ whenever

$$F(x) = 0,$$

for every $q - tuple$ of infinitesimal generators $v_1, \dots, v_q$ of G, then G is a q-symmetry group of the system.

As a special case for $q = 2$, we have generators v_1, v_2. The total vector field is the commutator $v = v_1 v_2 - v_2 v_1$. For $q > 2$ the total tensor $\mathrm{Alt}(v_1, \ldots, v_q)$ is not a vector field in general. We shall focus on the case $q > 2$ in the next section.

3 General equation for composition of symmetries

Theorem 3.1 (Characterization of q-symmetry group) *Let T linear operator on vector space V and let C the symmetry group of T. Then the q-symmetry group C_q is isomorphic to a power k of the symmetry group*

$$C_q \cong C^k,$$

for some $0 \leq k \leq q$. Moreover for matrix operators the maximal value $k = q$ is attained if and only if T is regular. In this case $H^q_{\mathrm{Sym}} \cong \mathbb{Z}$ for every $q \geq 0$.

Statement 3.1 *Let $v_1, \ldots, v_q$ vector fields. Then we have the recursive formula for generalized commutator*

$$[v_1, \ldots, v_q] = \sum_{k=0}^{q} (-1)^{1+k} [v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_q] v_k \tag{1}$$

For example for $q = 3$ we have

$$[A, B, C] = [A, B]C - [A, C]B + [B, C]A. \tag{2}$$

From the recursive formula we can see that $[v_1, \ldots, v_q]$ is a differential operator with components $\partial^k_{x_i}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, (q-1)$. For example for $q = 3$ and variables x, y, the generalized commutator $[v_1, v_2, v_3]$ has components $\partial_x, \partial_y, \partial^2_x, \partial^2_y$.

In general the equation for 2-symmetry is a system of first order partial differential equations given for $k = 1, \ldots, n$ as

$$\sum_{i=1}^{n} a^k(b^i_k) - b^k(a^i_k) = \alpha^k. \tag{3}$$

Moving on to 3-symmetry, we set the tensor

$$U^k_{ab} = a^k(b^x_k + b^y_k) - b^k(a^x_k + a^y_k) \tag{4}$$

and define the tensors of degree 3 as

$$U^{k,0}_{abc} = U^k_{ab} c^k - U^k_{ac} b^k + U^k_{bc} a^k, \tag{5}$$

and

$$U^{k,1}_{abc} = \sum_{l=0}^{n} U^k_{ab} c^k_l - U^k_{ac} b^k_l + U^k_{bc} a^k_l, \tag{6}$$

which gives the equations for 3-symmetry

In the general case we set inductively the tensors

$$U^{k,l}_{a^1 \dots a^{n+1}} = \sum_{i=1}^{n+1} \sum_{|\alpha|=0}^{l} (-1)^{1+i} U^k_{a^1 \dots a^{i-1} a^{i+1} \dots a^{n+1}} a^i_\alpha, \tag{7}$$

that gives the system of partial differential equations of $(q-1)$-the order for q-symmetry defined for $k = 1, \dots n$ and $l = 0, \dots (q-1)$

$$U^{k,l}_{a^1 \dots a^n} = \sum_{i=1}^{n} \sum_{|\alpha|=0}^{l} (-1)^{1+i} U^k_{a^1 \dots a^{i-1} a^{i+1} \dots a^n} a^i_\alpha. \tag{8}$$

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The general 2-symmetry equation for algebraic equations:

$$\begin{aligned} a^x b^x_x + a^y b^x_y - b^x a^x_x - b^y a^x_y &= a(x,y), \\ a^x b^y_x + a^y b^y_y - b^x a^y_x - b^y a^y_y &= b(x,y). \end{aligned} \tag{9}$$

First we will investigate the sphere. The equation is given by $F(x,y) = x^2 + y^2 - 1$. For the vector field

$$v = a\partial_x + b\partial_y, \tag{10}$$

we compute by the infinitesimal symmetry criterion

$$vF(x,y) = 2xa + 2yb = 0. \tag{11}$$

The solution is the generator $a = y, b = -x$. Integrating this vector field gives the symmetry group $SO(2)$. Next for 2-symmetry, we set the vector fields

$$\begin{aligned} v_1 &= a^x \partial_x + a^y \partial_y \\ v_2 &= b^x \partial_x + b^y \partial_y \end{aligned} \tag{12}$$

and solve for a^x, a^y, b^x, b^y the equation $v = [v_1, v_2]$. This gives the system of partial differential equations

$$\begin{aligned} a^x(b^x_x + b^y_x) - b^x(a^x_x + a^y_x) &= y, \\ a^y(b^x_y + b^y_y) - b^y(a^x_y + a^y_y) &= -x. \end{aligned} \tag{13}$$

We can set

$$\begin{aligned} a^x(x,y) &= c_1 + c_2x + c_3y + c_4xy, \\ a^y(x,y) &= c_5 + c_6x + c_7y + c_8xy, \\ b^x(x,y) &= c_9 + c_{10}x + c_{11}y + c_{12}xy, \\ b^y(x,y) &= c_{13} + c_{14}x + c_{15}y + c_{16}xy. \end{aligned} \tag{14}$$

so we have

$$b^x_x + b^y_x = c_{10} + c_{13} + (c_{14} + c_{16})y \tag{15}$$

and

$$\begin{aligned} a^x(b^x_x + b^y_x) = c_1(c_{10} + c_{13}) + (c_1(c_{14} + c_{16}) + c_3(c_{10} + c_{13}))y \\ +c_2(c_{10} + c_{13})x + (c_2(c_{14} + c_{16}) + c_4(c_{10} + c_{13}))xy \\ +c_3(c_{14} + c_{16})y^2 + c_4(c_{14} + c_{16})xy^2. \end{aligned} \tag{16}$$

This gives the first part of the system of equations

$$\begin{aligned} c_1(c_{10} + c_{13}) + (c_1(c_{14} + c_{16}) - c_5(c_2 + c_5) + (c_5(c_6 + c_8) = 0, \\ c_2(c_{10} + c_{13}) - c_{10}(c_2 + c_5) = 0, \\ c_1(c_{14} + c_{16}) + c_3(c_{10} + c_{13}) - c_9(c_6 + c_8) + c_{11}(c_2 + c_5) = 1, \\ c_2(c_{14} + c_{16}) + c_4(c_{10} + c_{13}) - c_{10}(c_6 + c_8) + c_{12}(c_2 + c_5), = 0 \\ c_3(c_{14} + c_{16}) - c_{11}(c_6 + c_8) = 0, \\ c_4(c_{14} + c_{16}) - c_{12}(c_6 + c_8) = 0. \end{aligned} \tag{17}$$

The second part of the system is

$$\begin{aligned} c_5(c_{14} + c_1) + (c_5(c_2 + c_4) - c_9(c_6 + c_{11}) + (c_9(c_{10} + c_{12}) = 0, \\ c_4(c_{14} + c_1) - c_{14}(c_6 + c_9) = 0, \\ c_4(c_2 + c_4) + c_7(c_{14} + c_1) - c_{13}(c_{10} + c_{12}) + c_{15}(c_5 + c_9) = -1, \\ c_6(c_2 + c_4) + c_8(c_{14} + c_1) - c_{14}(c_{10} + c_{12}) + c_{16}(c_6 + c_{11}), = 0 \\ c_7(c_2 + c_4) - c_{15}(c_{10} + c_{12}) = 0, \\ c_8(c_2 + c_4) - c_{16}(c_{10} + c_{12}) = 0. \end{aligned} \tag{18}$$

Since there are remaining 4 degrees of freedom, we see that the 2-symmetry group is isomorphic to the fourth power of the symmetry group $C_2 \cong C^4$.

For the hyperbola $F(x,y) = x^2 - y^2 + 1$ the generator is $a = y, b = x$, giving the group of hyperbolic rotations. The equation for 2-symmetry is in this case complementary to the sphere

$$\begin{aligned} a^x(b^x_x + b^y_x) - b^x(a^x_x + a^y_x) = y, \\ b^x(b^x_y + b^y_y) - b^y(a^x_y + a^y_y) = x. \end{aligned} \tag{19}$$

The system of equations is similar to (17) and (18) with 1 in the place of the only -1. Again we conclude that $C_2 \cong C^4$.

Our last example is the parabola given as $F(x, y) = \alpha x^2 - y$. We set the vector field v as above and compute

$$vF = 2\alpha x a - b = 0$$

so the solution generator is $a = 1, b = 2\alpha x$. Integrating this field gives the group of quadratic transformations. For the 2-symmetry equation we have

$$\begin{aligned} a^x(b^x_x + b^y_x) - b^x(a^x_x + a^y_x) &= 1, \\ a^y(b^x_y + b^y_y) - b^y(a^x_y + a^y_y) &= 2\alpha x. \end{aligned} \tag{20}$$

We can set

$$\begin{aligned} a^x(x, y) &= c_1 + c_2 x + c_4 xy, \\ a^y(x, y) &= c_5 + c_6 x + c_8 xy, \\ b^x(x, y) &= c_9 + c_{10} x + c_{12} xy, \\ b^y(x, y) &= c_{13} + c_{14} x + c_{16} xy. \end{aligned} \tag{21}$$

This yields the system given by

$$\begin{aligned} c_1(c_{10} + c_{13}) + (c_1(c_{14} + c_{16}) - c_5(c_2 + c_5) + (c_5(c_6 + c_8) &= 1, \\ c_2(c_{10} + c_{13}) - c_{10}(c_2 + c_5) &= 0, \\ c_1(c_{14} + c_{16}) - c_9(c_6 + c_8) + &= 0, \\ c_2(c_{14} + c_{16}) + c_4(c_{10} + c_{13}) - c_{10}(c_6 + c_8) + c_{12}(c_2 + c_5), &= 0 \\ c_4(c_{14} + c_{16}) - c_{12}(c_6 + c_8) &= 0, \end{aligned} \tag{22}$$

and

$$\begin{aligned} c_5(c_{14} + c_1) + (c_5(c_2 + c_4) - c_9(c_6) + (c_9(c_{10} + c_{12}) &= 0, \\ c_4(c_{14} + c_1) - c_{14}(c_6 + c_9) &= 2\alpha, \\ c_4(c_2 + c_4) - c_{13}(c_{10} + c_{12}) &= 0, \\ c_6(c_2 + c_4) + c_8(c_{14} + c_1) - c_{14}(c_{10} + c_{12}) + c_{16} c_6, &= 0 \\ c_8(c_2 + c_4) - c_{16}(c_{10} + c_{12}) &= 0. \end{aligned} \tag{23}$$

We can conclude that the 2-symmetry group is isomorphic to the second power of the symmetry group $C_2 \cong C^2$.

TODO HIGHER SYMMETRY TRICK example $q = 3$, by setting c^x, c^y to higher order partial derivative components ∂^l_x for $l > 0$ in TODO REF EQ, inductively returning to the $q = 2$ case. The coefficients for functions

a^x, b^x, a^y, b^y are given by the equations with partial derivatives of order 1 for $l = 0$ in TODO REF EQ with correction terms c_x^x, c_y^y.

TODO Magical power k Conjecture

For well-behaved linear operators all higher symmetry groups for $q \geq 2$ are expected to be the same power constant k power of the symmetry group $C_q \cong C^k$.

$$0 \to C \to C^k \to C^k \to \dots \tag{24}$$

DRAFT Suppose there exists k such that for every $q\ geq 2$, we have $C_q \cong C^k$. Then $H^q_{\mathrm{Sym}} \cong C^k$ for $q \geq 2$ and H^0_{Sym}.

4 Generalized symmetry criterion for ODE

TODO def total derivative

TODO def jet space, $M^{(n)}$, prolongation of vector field $\mathrm{pr}^{(n)}$

Statement 4.1 *Let M be an open subset of $X \times U$ and suppose $F(x, u^{(n)}) = 0$ is an n-th order system of differential equations defined over M, with corresponding subvariety $S \subset M^{(n)}$. Suppose G is a local group of transformations acting on M whose prolongation leaves S invariant, meaning whenever $(x, u^{(n)}) \in S$ we have $\mathrm{pr}^{(n)} g \cdot (x, u^{(n)}) \in S$ for all $g \in G$ such that this is defined. Then G is a symmetry group of the system of differential equations.*

Statement 4.2 *Let M be an open subset of $X \times U$ and suppose $F(x, u, \dots, u^{(n)}) = 0$ is an n-th order system of differential equations defined over M with corresponding subvariety $S \subset M^{(n)}$. Suppose G is a local group of transformations acting on M whose prolongation leaves S invariant, meaning that whenever $(x, u, \dots, u^{(n)} \in S$, we have $\mathrm{pr}^{(n)} g \cdot (x, u, \dots, u^{(n)}) \in S$ for all $g \in G$ such that this is defined. Then G is a symmetry group of the system of differential equations.*

Now using 4.2 we can formulate the infinitesimal symmetry criterion for ordinary differential equations.

Statement 4.3 (Infinitesimal invariance criterion for ODE) *Suppose*

$$F^k(x, u, \dots, u^{(n)}) = 0$$

for $k = 1, \dots, d$ is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M and

$$\mathrm{pr}^{(n)} v F^k(x, u, \dots, u^{(n)}) = 0,$$

where $k = 1, \dots, d$ whenever

$$F(x, u, \dots, u^{(n)}) = 0,$$

for every infinitesimal generator v of G, then G is a symmetry group of the system.

Theorem 4.1 (Generalized infinitesimal invariance criterion for ODE) *Suppose*

$$F^k(x, u, \dots, u^{(n)}) = 0$$

for $k = 1, \dots, d$ *is a system of differential equations of maximal rank defined over* $M \subset X \times U$. *If* G *is a local group of transformations acting on* M *and*

$$\mathrm{pr}^{(n)} v F^k(x, u, \dots, u^{(n)}) = 0,$$

where $k = 1, \dots, d$ *and* $v = \mathrm{Alt}(v_1, \dots, v_q)$ *whenever*

$$F(x, u, \dots, u^{(n)}) = 0,$$

for every $q-$ *tuple of infinitesimal generators* $v_1, \dots, v_q$ *of* G*, then* G *is a* q*-symmetry group of the system.*

Statement 4.4 *Let* F *be a system of differential equations of maximal rank defined over* $M \subset X \times U$. *The set of all infinitesimal symmetries of this system forms a Lie algebra of vector fields on* M. *Moreover, if this is finite-dimensional, the (connected component of the) symmetry group of the system is a local Lie group of transformations acting on* M.

Definition 4.1 (Characteristic of vector field) *The characteristic of the vector field* v *is a* q*-tuple of functions* $Q(x, u, u')$*, depending on* x, u *and first order derivatives of* u*, is defined by*

$$Q^\alpha(x, u, u_x) = b^\alpha(x, u) - \sum_{i=0}^{p} a^i(x, u) \frac{\partial u^\alpha}{\partial x^i} u_x,$$

for $\alpha = 1, \dots, q$.

Statement 4.5 *Let* $v = \sum_{i=1}^p a^i(x, u)\partial_{x_i} + \sum_{j=1}^t b^j(x, u)\partial_{u^j}$ *and let* Q *be its characteristic. The* n*-th prolongation of* v *is given as*

$$\mathrm{pr}^{(n)} v = \sum_{i=1}^{p} a^i(x, u)\partial_{x_i} + \sum_{j=1}^{t} \sum_{|\alpha|=0}^{n} b^j_\alpha(x, u^{(\alpha)})\partial^j_{u_\alpha} u_\alpha, \tag{25}$$

with coefficients

$$b^j_\alpha = D_\alpha Q^j + \sum_{i=0}^{p} a^i u^j_{\alpha,i}. \tag{26}$$

Our first example is the equation $u' = 0$, that is $u_x = 0$. We compute the first prolongation using the characteristic of vector field $v = a\partial_x + b\partial_u$ and using the infinitesimal symmetry criterion we obtain the equation $b_x = 0$. So the symmetry algebra is given by $a(x, u)\partial_x + b(u)\partial_u$. We move on to the equation of 2-symmetry given as

$$\begin{aligned} a^x(b^x_x+b^u_x)-b^x(a^x_x+a^u_x)&=a(x,u),\\ a^u(b^x_u+b^u_u)-b^u(a^x_u+a^u_u)&=b(u). \end{aligned}\tag{27}$$

The solution are functions $a^x(x,u), a^u(u), b^x(x,u), b^u(u)$ so we the 2-symmetry group of $u' = 0$ is isomorphic to the second power of the symmetry group $C_2 \cong C^2$.

Our second example is the second order equation $u'' = 0$. This yields by the infinitesimal symmetry criterion the system of equations

$$\begin{aligned} b_{xx}&=0,\\ 2b_{xu}&=a_{xx},\\ b_{uu}&=2a_{xu},\\ a_{uu}&=0. \end{aligned}\tag{28}$$

The Lie algebra of infinitesimal symmetries is isomorphic to $sl(3)$ with generators

$$\begin{aligned} v_1&=\partial_x,\\ v_2&=\partial_u,\\ v_3&=x\partial_x,\\ v_4&=u\partial_u,\\ v_5&=u\partial_x,\\ v_6&=x\partial_u,\\ v_7&=x^2\partial_x+xu\partial_u,\\ v_8&=xu\partial_x+u^2\partial_u. \end{aligned}\tag{29}$$

Now the equation of 2-symmetry for each generator $k = 1, \ldots, 8$ as $v_k = v^x_k\partial_x + v^u_k\partial_u$ is as follows

$$\begin{aligned} a^x(b^x_x+b^u_x)-b^x(a^x_x+a^u_x)&=v^x_k,\\ a^u(b^x_u+b^u_u)-b^u(a^x_u+a^u_u)&=v^y_k. \end{aligned}\tag{30}$$

So in conclusion $C_2 \cong C^3$.

Investigating the n-th order equation $u^{(n)} = 0$ we get from the infinitesimal symmetry criterion the system of equations

$$\begin{aligned} b_{x^n}&=0,\\ \binom{n}{k}b_{x^{k-1}u^{n+1-k}}&=a_{x^ku^{n-k}},\\ a_{u^n}&=0. \end{aligned}\tag{31}$$

for $k = 1, \ldots (n-1)$. we can set

$$\begin{aligned} a(x,u) &= a_0(x) + a_1(x)u + \ldots a_{n-1}(x)u^{n-1}, \\ b(x,u) &= b_0(x) + b_1(x)x + \ldots b_{n-1}(x)x^{n-1}. \end{aligned} \tag{32}$$

so we have $2n-2$ degrees of freedom and $n-4$ equations. This gives a symmetry algebra of dimension

5 Generalized symmetry criterion for PDE

First we restate the infinitesimal symmetry criterion for partial differential equations.

Theorem 5.1 (Generalized infinitesimal invariance criterion for PDE) *Suppose*

$$F^k(x_1, \ldots, x_d, u, \ldots, u^{(n)}) = 0$$

for $k = 1, \ldots, d$ is a locally solvable system of partial differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M and

$$\mathrm{pr}^{(n)} v F^k(x_1, \ldots, x_d, u, \ldots, u^{(n)}) = 0,$$

where $k = 1, \ldots, d$ and $v = \mathrm{Alt}(v_1, \ldots, v_q)$ whenever

$$F(x_1, \ldots, x_d, u, \ldots, u^{(n)}) = 0,$$

for every $q-tuple$ of infinitesimal generators $v_1, \ldots, v_q$ of G, then G is a q-symmetry group of the system.

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