Cohomology of higher symmetries

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* Dedicated to Maria Tůmová.

1 Introduction

The symmetry group of an equation is the group of transformations on the ambient space preserving the solution set. The aim of this paper is to generalize the notion of symmetry of an equation to higher symmetries and to provide basic results about structure of these higher symmetries. In particular we construct a cohomology based on symmetries and show that this cohomology is equivalent to singular cohomology of the submanifold given by the solution set in the ambient manifold. This procedure works generally for any type of equation. We also derive consequences from homology axioms, providing a general framework that generalizes the usual theorems for symmetries of partial and ordinary differential equations.

2 Symmetry of linear operator

First, we define the symmetry relation of a linear operator and compare it with the usual notion of symmetries of an equation.

Let T, A, B linear operators on vector space V over field k;

We will denote (matrix) multiplication of A and B by omitting the operation symbol as AB. In case this linear operator acts on a vector x, we will denote this as $(AB)x = ABx$, usually omitting the braces as the operation order is obvious from arity. In contrast, we shall denote group action by a linear operator A on another linear operator B at a point (vector) x defined by $A * B(x) = B(Ax)$, where Ax denotes matrix multiplication, in this case the same as plugging the point x into the linear operator B.

Definition 2.1 (Symmetry of linear operator) *Define symmetry of linear operator T relation $\sim$ for linear operators A, B such that $A \sim B$ whenever*

$$AT + TB = 0. \tag{1}$$

Definition 2.2 (Equivalence of symmetries) *Let T linear operator on vector space V. Let $[A_1, B_1]$ and $[A_2, B_2]$ symmetries of T. We say that the symmetries $[A_1, B_1], [A_2, B_2]$ are equivalent $[A_1, B_1] \sim [A_2, B_2]$ when there exist regular linear operators C_1, C_2 such that $A_2 = C_1 A_1 C_1^{-1}$ and $B_2 = C_2 B C_2^{-1}$.*

For vector space V of dimension n we can set B to be dependent on T and A since (1) has nullity at least n^2. This way we can solve the equation for A, B expressing B using A.

Statement 2.1 *Equivalence of symmetries of linear operator T relation $\sim$ is an equivalence relation.*

Proof 2.1 *Assume symmetry $[A, B]$. It is trivially equivalent to itself, establishing reflexivity. For another symmetry $[A', B']$ equivalent to $[A, B]$, we have a regular transformation J. Its inverse T^{-1} gives the transformation for the reversed relation, showing relation symmetry. Now for transitivity assume symmetry $[A", B"]$ equivalent to $[A', B']$ with regular transformation K. Then the composed transformation KJ shows similarity of $[A, B]$ and $[A", B"]$.*

For an element x in the solution set of $T = 0$ and a symmetry $[A, B]$ of T we have that

$$T(-Bx) = -(TB)x = A(Tx) = 0 \tag{2}$$

and so $-B$ preserves the solution set, being the usual point transformation as expected. For now, we define the symmetry group of a linear operator using the symmetry relation directly as in (2).

Definition 2.3 (Symmetry group) *Let T linear operator on vector space V with kernel X. The group G of all transformations $B : V \to V$ such that for every $x \in X$ we have $Bx \in X$.*

The easiest example of symmetry can be seen in a 2×2 matrix. Lets start with

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solving the equation (1) for a symmetry relation $[A, B]$ where B is expressed using A, we acquire the following solutions:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, B = \begin{bmatrix} a_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix}.$$

where $a_{11}, a_{12}, a_{22}, b_{21}, b_{22}$ are parameters, the last four are free. We can see that these transformations are related to translation by y, viewing the kernel of T as $\{[0, y] | y \in \mathbb{R}\}$.

As another example, we solve for symmetries of the linear operator

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The symmetry relation $[A, B]$ is given by generators $A_{11}, A_{21}, A_{12}, A_{22}$ and corresponding generators $B_{11}, B_{21}, B_{12}, B_{22}$:

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, B_{21} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, B_{12} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, B_{22} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since in the second case the matrix T is regular, there are no free parameters.

Having defined the usual symmetry relation, we move on to define the higher symmetries. We begin by defining the antisymmetrizer and insertion operator in order to keep the notation concise.

Definition 2.4 (Antisymmetrizer) *Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ linear operators on V. We define the antisymmetrizer of $A_1, \ldots, A_n$ as*

$$\mathrm{Alt}(A_1, \ldots, A_n) = \frac{1}{n!} \sum_{\rho \in S_n} sign(\rho) A_{\rho(1)} \ldots A_{\rho(n)}, \tag{3}$$

where S_n is the symmetric group of order n.

Statement 2.2 *Let $n \in \mathbb{N}$, $\rho \in S_n$, where S_n is the symmetric group of order n and $A_1, \ldots, A_n$ linear operators on V. Then (1) $\mathrm{Alt}(A_1, \ldots, A_n) = \mathrm{Alt}(\rho(A_1), \ldots, \rho(A_n))$ and as a corollary, we see that (2) $\mathrm{Alt}(A_1, \ldots, A_n) = 0$ when $A_i = A_j$ for some i and j.*

Proof 2.2 *(1): Obvious from definition* (3),
(2): From (A) we have $\mathrm{Alt}(A, B, B) = \frac{1}{3!}(ABB - ABB - BAB + ABA + BBA - BBA) = 0$. *Similar computation for $n > 3$.*

Definition 2.5 (Insertion operator) *Let $T, A_1, \dots, A_n$ linear operators on V and $1 \le k \le n$. We define the insertion operator for T as $i_T^k(A_1, \dots, A_n) = A_1 \dots A_{k-1} T A_{k+1} \dots A_n$. That is, the k-th term is replaced by T.*

Now we can define the higher symmetries of a linear operator using the antisymmetrizer and insertion operator.

Definition 2.6 (n-symmetry of linear operator) *Let $n \in \mathbb{N}$ and linear operators $A_1, \dots, A_{n+1}$ on V. We say that $[A_1, \dots, A_{n+1}]$ is an n-symmetry of T whenever the n-symmetry equation is satisfied*

$$\sum_{k=0}^{n} \mathrm{Alt}^k(i_T^k(A_{\rho(1)} \dots A_{\rho(n+1)})) = 0, \tag{4}$$

where Alt^k denotes the antisymmetrizer Alt acting on indices $1, \dots, k-1, k+1, \dots, n+1$ and fixing the k-th term.

The definition agrees with the usual definition of symmetry of linear operator 2.1 as above for $n = 1$. We shall denote left hand side of the equation (4) as $\mathrm{Sym}^n_{A_1,\dots A_n}(T)$ and the space of n-symmetries of T as $\mathrm{Sym}^n(T)$. We shall also denote $\mathrm{Sym}(T) = \bigcup\limits_{n=1}^{\infty} \mathrm{Sym}^n(T)$.

Definition 2.7 (Equivalence of symmetries) *Let T linear operator on vector space V. For $n \in \mathbb{N}$ let $[A_1, \dots, A_{n+1}]$ and $[B_1, \dots, B_{n+1}]$ be n-symmetries of T. We say that the n-symmetries $[A_1, \dots, A_{n+1}], [B_1, \dots, B_{n+1}]$ are equivalent $[A_1, \dots, A_{n+1}] \sim [B_1, \dots, B_{n+1}]$ when there exist regular linear operators $[C_1, \dots, C_{n+1}]$ such that $B_k = C_k A_k {C_k}^{-1}$ for $k = 1, \dots, n+1$.*

Statement 2.3 *For every $n \in \mathbb{N}$, n-symmetry of linear operator T relations $\sim$ are equivalence relations.*

Proof 2.3 *Similar to 2.1.*

Generally for vector space V of dimension n, for a k-symmetry $[A_1, \dots, A_{k+1}$, the equation (4) has nullity at least n^2 out of $(k+1)n^2$. So solving the equation, we can always express A_{k+1} using $A_1, \dots, A_k$.

Statement 2.4 (Symmetries form a vector space) *Let T linear operator. For every $n \in \mathbb{N}$, $\mathrm{Sym}^n(T)$, the n-symmetries of T form a vector space.*

Proof 2.4 *Let $[A, B]$ and $[A', B']$ symmetries. Then we have $(A + A')T = TB + TB' = T(B + B')$. For $\alpha \in \mathbb{R}$ we get $\alpha AT = \alpha BT$. The unit $[1, 1]$ is trivially a symmetry and so is the zero element $[0, 0]$.*

Again, let $n \in \mathbb{N}$, x an element in the solution set of $T = 0$ and a n-symmetry $[A_1, \dots, A_{n+1}]$ of T.

$$\mathrm{Sym}^n_{A_1,\dots,A_{n+1}}(T)x - \mathrm{Alt}(A_1,\dots,A_n)Tx = \mathrm{Alt}(A_1,\dots,A_n)T = 0 \tag{5}$$

giving the point transformation. For example for $n = 2$ and 2-symmetry $[A, B, C]$ the n-symmetry equation becomes

$$TBC - TCB + ATC - CTA + ABT - BAT = 0 \tag{6}$$

and so for a solution x of the equation $Tx = 0$ we have the point transformation symmetry given by

$$(TCB - TBC + ATC - CTA)x = [A, B]x = (ABT - BAT)x = 0, \tag{7}$$

where $[A, B]$ is the commutator of A and B. Generally, for $n \in \mathbb{N}$ the symmetry point transformation is given by $\mathrm{Sym}^n_{A_1,\dots,A_{n+1}}(T))x - \mathrm{Alt}(A_1,\dots,A_n)Tx$.

As a first example of higher symmetry, we will find 2-symmetries of the linear operator

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solving for the symmetry relation $[A, B, C]$, we obtain the following results:
We have triples of generators $A_{11}, A_{21}, A_{12}, A_{22}, B_{11}, B_{21}, B_{12}, B_{22}, C_{11}, C_{21}, C_{12}, C_{22}$.
Here $A_{11}, A_{21}, A_{12}, A_{22}, B_{11}, B_{21}, B_{12}, B_{22}$ are the usual matrix basis and $C_{11}, C_{21}, C_{12}, C_{22}$ are given by

$$C_{11} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}, C_{21} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}, C_{12} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}, C_{22} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}.$$

TODO DEBT RESULTS

Second we find 2-symmetries for the linear operator

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We obtain the following symmetry relation $[A, B, C]$: We use the same setup for $A_{11}, A_{21}, A_{12}, A_{22}$, $B_{11}, B_{21}, B_{12}, B_{22}$ as in the previous example. Additionally, $C_{11}, C_{21}, C_{12}, C_{22}$ are given by

$$C_{11} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}, C_{21} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}, C_{12} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}, C_{22} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}.$$

TODO DEBT RESULTS

3 Construction of symmetry cohomology

We begin the construction of the symmetry cohomology by identifying $\mathrm{Sym}^n(T)$ with a subspace in $\wedge^n(V)$.

Definition 3.1 (Identification with symmetric tensor algebra) *Let $n \in \mathbb{N}$ and $[A_1, \dots, A_{n+1}] \in \mathrm{Sym}^n(T)$. We define the embedding into $\wedge^n(V)$, the exterior algebra of degree n over V as $F_n(A_1, \dots, A_{n+1}) = \mathrm{Alt}(A_1, \dots, A_n)$, omitting the $n+1$-th term, where $\mathrm{Alt}(A_1, \dots, A_n)$ denotes here the tensor anti-symmetrization $\mathrm{Alt}(A_1, \dots, A_n) = \frac{1}{n!} \sum_{\rho \in S_n} \mathrm{sign}(\rho) A_{\rho(1)} \otimes \dots \otimes A_{\rho(n)}$ and S_n is the symmetric group on n elements.*

The wedge product on $\mathrm{Sym}^{n+k}(T)$ induced from $\wedge^{n+k}(V)$ for elements $x = (A_1, \dots, A_n)$ and $(A_{n+1}, \dots, A_{n+k})$ is given by

$$x \wedge y = \frac{1}{(n+k)!} \sum_{\rho \in S_(n+k)} \mathrm{sign}(\rho) A_{\rho(1)} \otimes \dots \otimes A_{\rho(n)} \otimes A_{\rho(n+1)} \otimes \dots \otimes A_{\rho(n+k)} \quad (8)$$

,

where S_{n+k} is the symmetric group on $n+k$ elements.

Suppose we have an n-symmetry given by $x = [A_1, \dots, A_n, B_1]$ and a k-symmetry $y = [A_{n+1}, \dots, A_{n+k}, B_2]$. The resulting $(n+k)$-symmetry obtained by $x \wedge y$ is a relation $[A_1, \dots, A_{n+k}, B]$ with B defined such that (4) holds.

The space $\mathrm{Sym}^n(T)$ can be easily seen to be n-linear.

Next we define a boundary operator on $\mathrm{Sym}^n(T)$, making it into a dg-algebra.

Definition 3.2 (Differential) *Let T linear operator on vector space V. Then for $n \in \mathbb{N}$ we can define the mapping $d^n : \mathrm{Sym}^n(T) \to \mathrm{Sym}^{n+1}(T)$ with $d^n(\mathrm{Alt}(A_1, \dots, A_n)) = \mathrm{Alt}(A_1, \dots, A_n, 1)$, where $(A_1, \dots, A_n) \in \mathrm{Sym}^n(T)$ and 1 is the identity on V.*

Theorem 3.1 ($\mathrm{Sym}^n(T)$ forms a dg-algebra) *(1) The mapping d from 3.2 is a differential on $\mathrm{Sym}^n(T)$. In particular, it satisfies (2) $d^2 = 0$ and (3) $d(A \otimes B) = dA \otimes B + (-1)^{degA} A \otimes dB$.*

Proof 3.1 *(1) Let us check that for an n-symmetry $[A_1, \dots, A_n]$, $d^n([A_1, \dots, A_n])$ is an $(n+1)$-symmetry. In particular $d^n([A_1, \dots, A_n]) = [A_1, \dots, A_n, 1]$. We need to find the transformation A_{n+2} that completes the full $(n+1)$-symmetry relation $[A_1, \dots, A_n, 1, A_{n+2}]$ that satisfies the symmetry equation (4). To this end, we group the terms in equation (4) for $n+1$ to three groups. First, terms that contain $A_{n+1} = 1$ and A_{n+2}. Second, terms that do not contain A_{n+1} but contain A_{n+2}. And the last third group, terms that do not contain A_{n+2}. This group is equivalent to $\mathrm{Alt}(A_1, \dots, A_n)T$. The first and third group terms each cancel out TODO DEBT USE INDUCTION PRESUMPTION (*), leaving only the second group which equal to antisymmetrizer of $A_1, \dots, A_n, T, A_{n+2}$ which*

fixes the $n+1$-th component, T. The equation given by setting this antisymmetrizer to zero yields A_{n+2}, since we can express A_{n+2} using $A_1, \ldots, A_n$.
(2) Let $A \in V$. We compute $d(d(A)) = d(\mathrm{Alt}(A,1)) = d(A,1,1)$. By 2.2 (2) we have $d(A,1,1) = 0$.
(3) We have $d(A \otimes B) = A \otimes B \otimes 1 - 1 \otimes A \otimes B$. From the other side $d(A) \otimes B + A \otimes d(B) = A \otimes 1 \otimes B - 1 \otimes A \otimes B + A \otimes B \otimes 1 - A \otimes 1 \otimes B = A \otimes B \otimes 1 - 1 \otimes A \otimes B$, which proves the statement.

In particular for $n = 1$ and a 1-symmetry $[A, B]$ we have $d(A) = d^1(A) = \frac{1}{2}(A \otimes 1 - 1 \otimes A)$.

The full 2-symmetry relation in this case is $[A, 1, C]$ where C is given by the equation $ATC = CTA$.

Now we may define the symmetry cohomology on $\mathrm{Sym}(T)$ as usual.

Definition 3.3 (Symmetry cohomology) *Let T linear operator on vector space V. Then for $k \geq 0$ the k-th symmetry cohomology group of T is defined as $H^k_{\mathrm{Sym}}(T) = \ker(d^k)/\mathrm{im}(d^{k-1})$, for d the differential defined in 3.2.*

4 A_∞ algebra structure on n-symmetries

First we define the notion of A_∞ algebra in order to show that the algebra of symmetries carries this structure.

Definition 4.1 (A-infinity algebra) *An A_∞ algebra is given by the triple [A, d, m_n], where A is a graded vector space $A = \bigcup_{n}^{\infty} A_n$ together with boundary operator $d : A_n \to A_{n+1}$ of degree 1 satisfying $d^2 = 0$, the pair [A, d] forms a dg-algebra and for every $n = 1, 2, \ldots$ we have a mapping $m_n : A^n \to A$ of degree $(2-n)$ that satisfies the Sheffield equation*

$$\sum_{u=0}^{n} (-1)^{r+st} m_u (1^{\otimes r} \otimes \cdots \otimes m_s \otimes 1^{\otimes t}) = 0; \tag{9}$$

where $n = r + s + t$ and $u = r + 1 + t$.

We will go over some examples of (9). For $n = 1$ the equation just shows m_1 is the differential d:

$$m_1 m_1 = 0. \tag{10}$$

For $n = 2$ the equation shows that m_2 is a graded derivation with respect to m_1:

$$m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1). \tag{11}$$

At $n = 3$ we have:

$$m_2(1 \otimes m_2 - m_2 \otimes 1) = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1). \tag{12}$$

Now we will show that the symmetries as defined in 3.1 form an A_∞ algebra.

Theorem 4.1 *Let k field, V vector space over k and $T : V \to V$ linear operator. Then symmetries A of T defined as in* 3.1 *form an A_∞ algebra together with the differential d from 3.2 as the boundary operator.*

Proof 4.1 *We already have that A is a dg-algebra by 3.1. It suffices to define the m_n operations.*

We proceed by defining $m_n(A_1 \otimes \cdots \otimes A_n) = \mathrm{Alt}(A_1, \ldots, A_n)$ and check that these operations satisfy the Sheffield equation (9)*.*

Let us sort the terms in the Sheffield equation into groups. First, terms of the form $m_1 m_n$. This term is equal to $\frac{1}{2n!}(d(\mathrm{Alt}(A_1, \ldots, A_n))$. This term comes first in the equation.

Second, terms of the form $m_n(m_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \ldots 1 \otimes m_1)$. The second term equals $\frac{1}{2n!}d(\mathrm{Alt}(A_1, \ldots, A_n))$. This terms comes last in the Sheffield equation.

Third, the middle terms. The rest of the terms that do not contain m_1 are of the form $m_a(m_b \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \ldots 1 \otimes m_b)$ for $a + b = n$. This term is equal to $\frac{n}{a!(b+1)!}\mathrm{Alt}(A_1, \ldots, A_n)$ when $a \neq b$, otherwise zero. These terms are antisymmetric with respect to n.

Summing all the terms we see that the first and second terms cancel out as well as the middle terms so the Sheffield equation holds.

Having established that symmetries form an A_∞ algebra, We make use of Kadeischvili Theorem as in [4] in order to induce the A_∞ algebra structure on $H^k_{\mathrm{Sym}}(T)$.

Definition 4.2 (Quasi-isomorphism) *We say that two A_∞ algebras are quasi-isomorphic, whenever their cohomology rings are isomorphic.*

Definition 4.3 (A_∞ algebra morphism) *A morphism of A_∞ algebras $f : A \to B$ is a family $f_n : A^{\otimes n} \to B$ of graded maps of degree $1 - n$ such that*

$$\sum (-1)^{r+st} f_u(1^{\otimes r} \otimes m_s \otimes a^{\otimes t}) = \sum (-1)^s m_r(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_r}), \qquad (13)$$

where $n \geq 0$, $n = r + s + t$, $u = r + 1 + t$ and the second sum runs over all $1 \leq r \leq n$ and all decompositions $n = i_1 + i_2 + \cdots + i_r$. The sign on the right hand side of the equation is given by

$$s = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \cdots + 2(i_{r-2} - 1) + (i_{r-1} - 1).$$

Moreover f is a quasi-isomorphism if f_1 is a quasi-isomorphism.

Theorem 4.2 (Kadeishvili Theorem) *Let A an A_∞ algebra and let $H^*(A)$ the cohomology of A. There is an A_∞ algebra structure on $H^*(A)$ with $m_1 = 0$ and m_2 induced by multiplication, constructed from the A_∞-structure of A, such that there is a quasi-isomorphism of A_∞ algebras $H^*(A) \to A$, lifting the identity of $H^*(A)$. This A_∞-algebra structure is unique up to quasi-isomorphism.*

5 Calculation of higher symmetry groups - Algebraic equations

Here the simplified notation a_u means first order partial derivative of a with respect to u.

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Theorem 5.1 (Characterization of q-symmetry group) *Let T linear operator on vector space V and let C the symmetry group of T. Then the q-symmetry group C_q is isomorphic to a power k of the symmetry group*

$$C_q \cong C^k,$$

for some $0 \leq k \leq q$. Moreover for finite-dimensional operators the maximal value $k = q$ is attained if and only if T is regular. In this case $H^q_{\text{Sym}} \cong \mathbb{Z}$ for every $q \geq 0$.

TODO def invariant function

TODO def infinitesimal generator, symmetry algebra

Statement 5.1 *Let G be a connected group of transformations acting on the manifold M. A smooth real-valued function $f : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$v(f) = 0,$$

for all $x \in M$ and every infinitesimal generator v of G.

Statement 5.2 *Let M be an open subset of $X \times U$ and suppose $F(x, u^{(n)}) = 0$ is an n-th order system of differential equations defined over M, with corresponding subvariety $S \subset M^{(n)}$. Suppose G is a local group of transformations acting on M whose prolongation leaves S invariant, meaning whenever $(x, u^{(n)}) \in S$ we have $\text{pr}^{(n)} g \cdot (x, u^{(n)}) \in S$ for all $g \in G$ such that this is defined. Then G is a symmetry group of the system of differential equations.*

Statement 5.3 (Infinitesimal invariance criterion for algebraic equations) *Suppose*

$$F^k(x) = 0$$

for $k = 1, \ldots, d$ is a system of algebraic equations of maximal rank defined over M. If G is a connected local Lie group of transformations acting on M then G is a symmetry group of the system if and only if

$$vF^k(x) = 0,$$

where $k = 1, \ldots, d$ whenever

$$F(x) = 0,$$

for every infinitesimal generator v of G,

Theorem 5.2 (Generalized infinitesimal invariance criterion for algebraic equations) *Let G be a connected Lie group of transformations acting on the n-dimensional manifold M. Let $F : M \to \mathbb{R}^d$, $d \leq n$ define a system of algebraic equations*

$$F^k(x) = 0$$

for $k = 1, \ldots, d$ and assume the system is of maximal rank, meaning the Jacobian matrix $(\frac{\partial F^k}{\partial x_i})$ is of rank d at every solution x of the system. Then G is a q-symmetry group of the system if and only if

$$vF^k(x) = 0,$$

where $k = 1, \ldots, d$ and $v = \mathrm{Alt}(v_1, \ldots, v_q)$ whenever

$$F(x) = 0,$$

for every $q-$ tuple of infinitesimal generators $v_1, \ldots, v_q$ of G, then G is a q-symmetry group of the system.

As a special case for $q = 2$, we have generators v_1, v_2. The total vector field is the commutator $v = v_1 v_2 - v_2 v_1$. For $q > 2$ the total tensor $\mathrm{Alt}(v_1, \ldots, v_q)$ is not a vector field in general. We shall denote this generalized commutator of vector fields as $[v_1, \ldots, v_q]$.

Statement 5.4 *Let $v_1, \ldots, v_q$ vector fields. Then we have the recursive formula for generalized commutator*

$$[v_1, \ldots, v_q] = \sum_{k=0}^{q} (-1)^{1+k} [v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_q] v_k \tag{14}$$

For example for $q = 3$ we have

$$[A, B, C] = [A, B]C - [A, C]B + [B, C]A. \tag{15}$$

From the recursive formula we can see that $[v_1, \ldots, v_q]$ is a differential operator with components $\partial_{x_i}^k$ for $i = 1, \ldots, n$ and $k = 1, \ldots, (q-1)$. For example for $q = 3$ and variables x, y, the generalized commutator $[v_1, v_2, v_3]$ has components $\partial_x, \partial_y, \partial_x^2, \partial_y^2$.

First we will investigate the sphere. The equation is given by $F(x, y) = x^2 + y^2 - 1$. For the vector field

$$v = a\partial_x + b\partial_y, \tag{16}$$

we compute by the infinitesimal symmetry criterion

$$vF(x, y) = 2xa + 2yb = 0. \tag{17}$$

The solution is the generator $a = y, b = -x$. Integrating this vector field gives the symmetry group $SO(2)$. Next for 2-symmetry, we set the vector fields

$$\begin{aligned} v_1 &= a^x \partial_x + a^y \partial_y \\ v_2 &= b^x \partial_x + b^y \partial_y \end{aligned} \tag{18}$$

and solve for a^x, a^y, b^x, b^y the equation $v = [v_1, v_2]$. This gives the system of partial differential equations

$$\begin{aligned} a^x(b^x_x + b^y_x) - b^x(a^x_x + a^y_x) &= y, \\ a^y(b^x_y + b^y_y) - b^y(a^x_y + a^y_y) &= -x. \end{aligned} \tag{19}$$

We can set

$$\begin{aligned} a^x(x,y) &= c_1 + c_2 x + c_3 y + c_4 xy, \\ a^y(x,y) &= c_5 + c_6 x + c_7 y + c_8 xy, \\ b^x(x,y) &= c_9 + c_{10} x + c_{11} y + c_{12} xy, \\ b^y(x,y) &= c_{13} + c_{14} x + c_{15} y + c_{16} xy. \end{aligned} \tag{20}$$

so we have

$$b^x_x + b^y_x = c_{10} + c_{13} + (c_{14} + c_{16})y \tag{21}$$

and

$$\begin{aligned} a^x(b^x_x + b^y_x) = c_1(c_{10} + c_{13}) &+ (c_1(c_{14} + c_{16}) + c_3(c_{10} + c_{13}))y \\ &+ c_2(c_{10} + c_{13})x + (c_2(c_{14} + c_{16}) + c_4(c_{10} + c_{13}))xy \\ &+ c_3(c_{14} + c_{16})y^2 + c_4(c_{14} + c_{16})xy^2. \end{aligned} \tag{22}$$

This gives the first part of the system of equations

$$\begin{aligned} c_1(c_{10} + c_{13}) + (c_1(c_{14} + c_{16}) - c_5(c_2 + c_5) + (c_5(c_6 + c_8) &= 0, \\ c_2(c_{10} + c_{13}) - c_{10}(c_2 + c_5) &= 0, \\ c_1(c_{14} + c_{16}) + c_3(c_{10} + c_{13}) - c_9(c_6 + c_8) + c_{11}(c_2 + c_5) &= 1, \\ c_2(c_{14} + c_{16}) + c_4(c_{10} + c_{13}) - c_{10}(c_6 + c_8) + c_{12}(c_2 + c_5), &= 0 \\ c_3(c_{14} + c_{16}) - c_{11}(c_6 + c_8) &= 0, \\ c_4(c_{14} + c_{16}) - c_{12}(c_6 + c_8) &= 0. \end{aligned} \tag{23}$$

The second part of the system is

$$\begin{aligned} c_5(c_{14} + c_1) + (c_5(c_2 + c_4) - c_9(c_6 + c_{11}) + (c_9(c_{10} + c_{12}) &= 0, \\ c_4(c_{14} + c_1) - c_{14}(c_6 + c_9) &= 0, \\ c_4(c_2 + c_4) + c_7(c_{14} + c_1) - c_{13}(c_{10} + c_{12}) + c_{15}(c_5 + c_9) &= -1, \\ c_6(c_2 + c_4) + c_8(c_{14} + c_1) - c_{14}(c_{10} + c_{12}) + c_{16}(c_6 + c_{11}), &= 0 \\ c_7(c_2 + c_4) - c_{15}(c_{10} + c_{12}) &= 0, \\ c_8(c_2 + c_4) - c_{16}(c_{10} + c_{12}) &= 0. \end{aligned} \tag{24}$$

Since there are remaining 4 degrees of freedom, we see that the 2-symmetry group is isomorphic to the fourth power of the symmetry group $C_2 \cong C^4$.

For the hyperbola $F(x,y) = x^2 - y^2 + 1$ the generator is $a = y, b = x$, giving the group of hyperbolic rotations. The equation for 2-symmetry is in this case complementary to the sphere

$$\begin{aligned} a^x(b^x_x + b^y_x) - b^x(a^x_x + a^y_x) &= y, \\ b^x(b^x_y + b^y_y) - b^y(a^x_y + a^y_y) &= x. \end{aligned} \tag{25}$$

The system of equations is similar to (23) and (24) with 1 in the place of the only -1. Again we conclude that $C_2 \cong C^4$.

Our last example is the parabola given as $F(x,y) = \alpha x^2 - y$. We set the vector field v as above and compute

$$vF = 2\alpha x a - b = 0$$

so the solution generator is $a = 1, b = 2\alpha x$. Integrating this field gives the group of quadratic transformations. For the 2-symmetry equation we have

$$\begin{aligned} a^x(b^x_x + b^y_x) - b^x(a^x_x + a^y_x) &= 1, \\ a^y(b^x_y + b^y_y) - b^y(a^x_y + a^y_y) &= 2\alpha x. \end{aligned} \tag{26}$$

We can set

$$\begin{aligned} a^x(x,y) &= c_1 + c_2 x + c_4 xy, \\ a^y(x,y) &= c_5 + c_6 x + c_8 xy, \\ b^x(x,y) &= c_9 + c_{10} x + c_{12} xy, \\ b^y(x,y) &= c_{13} + c_{14} x + c_{16} xy. \end{aligned} \tag{27}$$

This yields the system given by

$$\begin{aligned} c_1(c_{10} + c_{13}) + (c_1(c_{14} + c_{16}) - c_5(c_2 + c_5) + (c_5(c_6 + c_8) &= 1, \\ c_2(c_{10} + c_{13}) - c_{10}(c_2 + c_5) &= 0, \\ c_1(c_{14} + c_{16}) - c_9(c_6 + c_8) + &= 0, \\ c_2(c_{14} + c_{16}) + c_4(c_{10} + c_{13}) - c_{10}(c_6 + c_8) + c_{12}(c_2 + c_5), &= 0 \\ c_4(c_{14} + c_{16}) - c_{12}(c_6 + c_8) &= 0, \end{aligned} \tag{28}$$

and

$$\begin{aligned} c_5(c_{14} + c_1) + (c_5(c_2 + c_4) - c_9(c_6) + (c_9(c_{10} + c_{12}) &= 0, \\ c_4(c_{14} + c_1) - c_{14}(c_6 + c_9) &= 2\alpha, \\ c_4(c_2 + c_4) - c_{13}(c_{10} + c_{12}) &= 0, \\ c_6(c_2 + c_4) + c_8(c_{14} + c_1) - c_{14}(c_{10} + c_{12}) + c_{16}c_6, &= 0 \\ c_8(c_2 + c_4) - c_{16}(c_{10} + c_{12}) &= 0. \end{aligned} \tag{29}$$

We can conclude that the 2-symmetry group is isomorphic to the second power of the symmetry group $C_2 \cong C^2$.

In general the equation for 2-symmetry is a system of first order partial differential equations given for $k = 1, \dots, n$ as

$$\sum_{i=1}^{n} a^k(b^i_k) - b^k(a^i_k) = \alpha^k. \tag{30}$$

Moving on to 3-symmetry, we set the tensor

$$U^k_{ab} = a^k(b^x_k + b^y_k) - b^k(a^x_k + a^y_k) \tag{31}$$

and define the tensors of degree 3 as

$$U^{k,0}_{abc} = U^k_{ab}c^k - U^k_{ac}b^k + U^k_{bc}a^k, \tag{32}$$

and

$$U^{k,1}_{abc} = \sum_{l=0}^{n} U^k_{ab}c^k_l - U^k_{ac}b^k_l + U^k_{bc}a^k_l, \tag{33}$$

which gives the equations for 3-symmetry

In the general case we set inductively the tensors

$$U^{k,l}_{a^1 \dots a^{n+1}} = \sum_{i=1}^{n+1} \sum_{|\alpha|=0}^{l} (-1)^{1+i} U^k_{a^1 \dots a^{i-1} a^{i+1} \dots a^{n+1}} a^i_\alpha, \tag{34}$$

that gives the system of partial differential equations of $(q-1)$-the order for q-symmetry defined for $k = 1, \dots n$ and $l = 0, \dots (q-1)$

$$U^{k,l}_{a^1 \dots a^n} = \sum_{i=1}^{n} \sum_{|\alpha|=0}^{l} (-1)^{1+i} U^k_{a^1 \dots a^{i-1} a^{i+1} \dots a^n} a^i_\alpha. \tag{35}$$

TODO Magical power k Conjecture

$$0 \to C \to C^k \to C^k \to \dots \tag{36}$$

DRAFT

The general 2-symmetry equation for algebraic equations:

$$\begin{aligned} a^x b^x_x + a^y b^x_y - b^x a^x_x - b^y a^x_y &= a(x,y), \\ a^x b^y_x + a^y b^y_y - b^x a^y_x - b^y a^y_y &= b(x,y). \end{aligned} \tag{37}$$

6 Calculation of higher symmetry groups - ODE

TODO def total derivative

TODO def jet space, $M^{(n)}$, prolongation of vector field $\mathrm{pr}^{(n)}$

Statement 6.1 *Let M be an open subset of $X \times U$ and suppose $F(x, u, \dots, u^{(n)}) = 0$ is an n-th order system of differential equations defined over M with corresponding subvariety $S \subset M^{(n)}$. Suppose G is a local group of transformations acting on M whose prolongation leaves S invariant, meaning that whenever $(x, u, \dots, u^{(n)} \in S$, we have $\mathrm{pr}^{(n)} g \cdot (x, u, \dots, u^{(n)}) \in S$ for all $g \in G$ such that this is defined. Then G is a symmetry group of the system of differential equations.*

Now using 6.1 we can formulate the infinitesimal symmetry criterion for ordinary differential equations.

Statement 6.2 (Infinitesimal invariance criterion for ODE) *Suppose*

$$F^k(x, u, \dots, u^{(n)}) = 0$$

for $k = 1, \dots, d$ is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M and

$$\mathrm{pr}^{(n)} v F^k(x, u, \dots, u^{(n)}) = 0,$$

where $k = 1, \dots, d$ whenever

$$F(x, u, \dots, u^{(n)}) = 0,$$

for every infinitesimal generator v of G, then G is a symmetry group of the system.

Theorem 6.1 (Generalized infinitesimal invariance criterion for ODE) *Suppose*

$$F^k(x, u, \dots, u^{(n)}) = 0$$

for $k = 1, \dots, d$ is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M and

$$\mathrm{pr}^{(n)} v F^k(x, u, \dots, u^{(n)}) = 0,$$

where $k = 1, \dots, d$ and $v = \mathrm{Alt}(v_1, \dots, v_q)$ whenever

$$F(x, u, \dots, u^{(n)}) = 0,$$

for every $q-$tuple of infinitesimal generators $v_1, \dots, v_q$ of G, then G is a q-symmetry group of the system.

Statement 6.3 *Let F be a system of differential equations of maximal rank defined over $M \subset X \times U$. The set of all infinitesimal symmetries of this system forms a Lie algebra of vector fields on M. Moreover, if this is finite-dimensional, the (connected component of the) symmetry group of the system is a local Lie group of transformations acting on M.*

Definition 6.1 (Characteristic of vector field) *The characteristic of the vector field v is a q-tuple of functions $Q(x, u, u')$, depending on x, u and first order derivatives of u, is defined by*

$$Q^\alpha(x, u, u_x) = b^\alpha(x, u) - \sum_{i=0}^{p} a^i(x, u)\frac{\partial u^\alpha}{\partial x^i} u_x,$$

for $\alpha = 1, \ldots, q$.

Statement 6.4 *Let $v = \sum_{i=1}^p a^i(x, u)\partial_{x_i} + \sum_{j=1}^t b^j(x, u)\partial_{u^j}$ and let Q be its characteristic. The n-th prolongation of v is given as*

$$\mathrm{pr}^{(n)}v = \sum_{i=1}^{p} a^i(x, u)\partial_{x_i} + \sum_{j=1}^{t}\sum_{|\alpha|=0}^{n} b^j_\alpha(x, u^{(\alpha)})\partial^j_{u_\alpha} u_\alpha, \tag{38}$$

with coefficients

$$b^j_\alpha = D_\alpha Q^j + \sum_{i=0}^{p} a^i u^j_{\alpha,i}. \tag{39}$$

Our first example is the equation $u' = 0$, that is $u_x = 0$. We compute the first prolongation using the characteristic of vector field $v = a\partial_x + b\partial_u$ and using the infinitesimal symmetry criterion we obtain the equation $b_x = 0$. So the symmetry algebra is given by $a(x, u)\partial_x + b(u)\partial_u$. We move on to the equation of 2-symmetry given as

$$\begin{aligned} a^x(b^x_x + b^u_x) - b^x(a^x_x + a^u_x) &= a(x, u), \\ a^u(b^x_u + b^u_u) - b^u(a^x_u + a^u_u) &= b(u). \end{aligned} \tag{40}$$

The solution are functions $a^x(x, u), a^u(u), b^x(x, u), b^u(u)$ so we the 2-symmetry group of $u' = 0$ is isomorphic to the second power of the symmetry group $C_2 \cong C^2$.

Our second example is the second order equation $u'' = 0$. This yields by the infinitesimal symmetry criterion the system of equations

$$\begin{aligned} b_{xx} &= 0, \\ 2b_{xu} &= a_{xx}, \\ b_{uu} &= 2a_{xu}, \\ a_{uu} &= 0. \end{aligned} \tag{41}$$

The Lie algebra of infinitesimal symmetries is isomorphic to $sl(3)$ with generators

$$\begin{aligned} v_1 &= \partial_x, \\ v_2 &= \partial_u, \\ v_3 &= x\partial_x, \\ v_4 &= u\partial_u, \\ v_5 &= u\partial_x, \\ v_6 &= x\partial_u, \\ v_7 &= x^2\partial_x + xu\partial_u, \\ v_8 &= xu\partial_x + u^2\partial_u. \end{aligned} \tag{42}$$

Now the equation of 2-symmetry for each generator $k = 1, \ldots, 8$ as $v_k = v_k^x\partial_x + v_k^u\partial_u$ is as follows

$$\begin{aligned} a^x(b_x^x + b_x^u) - b^x(a_x^x + a_x^u) = v_k^x, \\ a^u(b_u^x + b_u^u) - b^u(a_u^x + a_u^u) = v_k^y. \end{aligned} \tag{43}$$

So in conclusion $C_2 \cong C^3$.

Investigating the n-th order equation $u^{(n)} = 0$ we get from the infinitesimal symmetry criterion the system of equations

$$\begin{aligned} b_{x^n} &= 0, \\ \binom{n}{k} b_{x^{k-1}u^{n+1-k}} &= a_{x^k u^{n-k}}, \\ a_{u^n} &= 0. \end{aligned} \tag{44}$$

for $k = 1, \ldots (n-1)$. we can set

$$\begin{aligned} a(x,u) = a_0(x) + a_1(x)u + \ldots a_{n-1}(x)u^{n-1}, \\ b(x,u) = b_0(x) + b_1(x)x + \ldots b_{n-1}(x)x^{n-1}. \end{aligned} \tag{45}$$

so we have $2n - 2$ degrees of freedom and $n - 4$ equations. This gives a symmetry algebra of dimension

7 Calculation of higher symmetry groups - PDE

First we restate the infinitesimal symmetry criterion for partial differential equations.

Theorem 7.1 (Generalized infinitesimal invariance criterion for PDE) *Suppose*

$$F^k(x_1, \dots, x_d, u, \dots, u^{(n)}) = 0$$

for $k = 1, \dots, d$ is a locally solvable system of partial differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M and

$$\mathrm{pr}^{(n)} v F^k(x_1, \dots, x_d, u, \dots, u^{(n)}) = 0,$$

where $k = 1, \dots, d$ and $v = \mathrm{Alt}(v_1, \dots, v_q)$ whenever

$$F(x_1, \dots, x_d, u, \dots, u^{(n)}) = 0,$$

for every $q-$ tuple of infinitesimal generators $v_1, \dots, v_q$ of G, then G is a q-symmetry group of the system.

8 Consequences of homology axioms

Definition 8.1 (Equation) *TODO equation ambient space topological space / manifold continuous mapping / C-whatever solution set*

The solution set of an equation is always a topological subspace of the ambient space, since it is the continuous preimage of a one point set. In the case when the ambient space is a manifold, the solution set becomes a submanifold.

Definition 8.2 (Equivalent equations) *We say that two equations ϵ_1, ϵ_2 are equivalent, whenever using a solution of ϵ_1 we can deduce a solution for ϵ_2.*

Statement 8.1 *Relation of equivalence for equations is an equivalence relation.*

Definition 8.3 (Local transformation group) *TODO*

Definition 8.4 (Symmetry group of system of equations) *TODO*

Having established symmetry cohomology on symmetries of a linear operator T on a vector space V, we now derive general theorems on symmetries of equations from the axioms of homology.

Statement 8.2 (Eilenberg-Steenrod cohomology axioms) *Let H^n sequence of functors for $n \in \mathbb{N}$ from the category of pairs of topological spaces (X, A) to the category of graded commutative rings, together with the boundary map $\delta : H^i(X, A) \to H^{i+1}(A)$ which is a natural transformation. Then the following holds:*
(1) Homotopy: Homotopic maps induce the same map in homology. That is, if $g, h : (X, A) \to (Y, B)$ are homotopic, then their induced homomorphisms are the same.

(2) Excision: If (X, A) is a topological pair and U is a subset of A such that the closure of U is contained in the interior of A, then the inclusion map $i : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism in homology.
(3) Dimension: Let P be the one-point space, then $H^n(P) = 0$ for $n \neq 0$.
(4) Additivity: If $X = \bigcup_\alpha X_\alpha$, the disjoint union of a family of topological spaces X_α, then $H^n(X) \cong \bigoplus_\alpha H^n(X_\alpha)$
(5) Exactness: Each pair (X, A) induces a long exact sequence in homology, via the inclusions $i : A \to X$ and $j : X \to (X, A)$:

$$\cdots \to H^n(X, A) \to^{j_*} H^n(X) \to^{i_*} H^n(A) \to^{\delta} H^{n+1}(X, A) \to \dots \tag{46}$$

If P is the one-point space, then $H_0(P)$ is called the coefficient group.

TODO homology axioms list enumerate, rightarrow indices

Statement 8.3 *Let ϵ_1, ϵ_2 equivalent equations. Then for every $k \geq 0$, we have $H^k_{\mathrm{Sym}}(\epsilon_1) \cong H^k_{\mathrm{Sym}}(\epsilon_2)$.*

Proof 8.1 *Consequence of 8.2 (1).*

Statement 8.4 (Substitution Theorem) *Any two equations related by a change of coordinates have isomorphic symmetry cohomology groups.*

Proof 8.2 *Equations related by a change of coordinates are obviously equivalent.*

As an example we can see second-order linear partial differential equations. Lets say we have an equation ϵ of elliptic class, given by the quadratic form

$$T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

We can substitute variables, transforming ϵ to an equivalent equation ϵ' of the same class, given by

$$T' = \begin{bmatrix} A & B \\ B & C \end{bmatrix}.$$

We can conclude that second-order linear partial differential equations (and generally equations given by a quadratic form) of the same class have isomorphic symmetry homology.

Statement 8.5 (Dimension) *Any equation ϵ on one point space P has $H^n_{\mathrm{Sym}} = 0$ for $n \neq 0$ and H^0_{Sym} is the symmetry group of ϵ.*

Proof 8.3 *Consequence of 8.2 (3).*

Statement 8.6 (Additivity) *Let ϵ equation defined on $X = \bigcup_\alpha X_\alpha$. Denote ϵ_α the component of the equation ϵ defined on X_α. Then $H_{\text{Sym}}(\epsilon) \cong \bigoplus_\alpha H_{\text{Sym}}(\epsilon_\alpha)$.*

Proof 8.4 *Consequence of 8.2 (4).*

Statement 8.7 (Excision Theorem) *Let ϵ equation defined on space X with solution set S and subset U of S such that the closure of U is contained in the interior of S. Then for $k \geq 0$ we have $H^k_{\text{Sym}}(X \setminus U, S \setminus U) \cong H^k_{\text{Sym}}(X, S)$.*

Proof 8.5 *Consequence of 8.2 (2).*

Statement 8.8 (Long exact sequence) *Let ϵ equation defined on space X with solution set S. Then we have the following long exact sequence:*

$$\cdots \to H^n(X,S) \to H^n(X) \to H^n(S) \to H^{n+1}(X,S) \to \dots \tag{47}$$

Proof 8.6 *Consequence of 8.2 (5).*

Statement 8.9 (Mayer-Vietoris sequence) *Let X topological space with subspaces A, B whose interiors cover X. Then we have the following long exact sequence:*

$$\cdots \to H^n_{\text{Sym}}(X) \to H^n_{\text{Sym}}(A) \oplus H^n_{\text{Sym}}(B) \to H^n_{\text{Sym}}(A \cap B) \to H^{n+1}_{\text{Sym}}(X) \to \dots \tag{48}$$

Proof 8.7 *Follows from 8.2 (2) and (5).*

Statement 8.10 (Mayer-Vietoris sequence for equations) *Let ϵ_1, ϵ_2 equations with solution sets A and B. Then we have the following long exact sequence:*

$$\cdots \to H^n_{\text{Sym}}(A \cup B) \to H^n_{\text{Sym}}(A) \oplus H^n_{\text{Sym}}(B) \to H^n_{\text{Sym}}(A \cap B) \to H^{n+1}_{\text{Sym}}(A \cup B) \to \dots \tag{49}$$

Proof 8.8 *Consequence of 8.9.*

Definition 8.5 (Suspension) *Let X topological space. Then the suspension SX of X is defined as the quotient space $X \times [0,1]/X \times \{0\}, X \times \{1\}$.*

Statement 8.11 (Suspension Theorem) *Let $X = SY$ suspension of a topological space Y. Then $H^n(SY) \cong H^{n-1}(Y)$.*

Proof 8.9 *Follows from 8.9.*

As an example, consider the family of equations $\epsilon_n : X_1^2 + \cdots + X_n^2 = r^2$, for $r > 0$. The solution set is the topological space $\mathbb{S}^n$. We can apply suspension, increasing the degree of the equation $S(\mathbb{S}^n) \cong \mathbb{S}^{n+1}$. We can compute now using the Suspension Theorem that $H^k_{\mathrm{Sym}}(\mathbb{S}^n) \cong \mathbb{Z}$ for $k = 0, n$ and $H^k_{\mathrm{Sym}}(\mathbb{S}^n) \cong \{0\}$ otherwise.

Definition 8.6 (Solvable group) *A group G is solvable if there is a chain of subgroups $1 = G_1 \leq G_2 \leq \cdots \leq G_n = G$ such that for $k = 1, 2, \ldots, n-1$ we have that G_{k-1} is normal in G_k and G_k/G_{k-1} is abelian.*

We note that direct product of solvable groups is solvable. So every symmetry cohomology group is solvable if and only if the symmetry group is solvable.

Definition 8.7 (Solvable symmetry group) *TODO*

Definition 8.8 (Orbit) *TODO*

Statement 8.12 (Reduction Theorem) *Let ϵ n-th order equation that admits an r-parameter solvable group of symmetries G such that for $1 \leq k \leq r$ the orbits of G^k have dimension k. Then ϵ is equivalent to a $(n-r)$-th order equation.*

TODO GENERALIZE TO HIGHER SYMMETRIES

Proof 8.10 *TODO, hopefully using homology axioms*

TODO Example of usage of homology theorems

TODO Examples of equations with trivial cohomology, similar to spheres and such

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